Morita equivalences between fixed point algebras and crossed products

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Abstract

In this paper, we will prove that if A is a C^*-algebra with an effective coaction ϵ by a compact quantum group, then the fixed point algebra and the reduced crossed product are Morita equivalent. As an application, we prove an imprimitivity type theorem for crossed products of coactions by discrete Kac C^*-algebras.

0. Introduction

After proving an imprimitivity type theorem for multiplicative unitaries of discrete type (actually, those come from discrete Kac algebras) in [7, 5.2], we noticed that the same method can be use to prove the fact that the fixed point algebra and the reduced crossed product of an effective coaction (see Definition 2.2) on a C^*-algebra A by a Woronowicz C^*-algebra (i.e. a compact quantum group) are Morita equivalent (see Theorem 2.13). However, since A may not be unital, we need a slightly more general version of Watatani's C^*-basic construction. We do this in Section 1.

In Section 2, we prove the main theorem. We first prove this in the case when the coaction is injective and the Hopf C^*-algebra is the reduced one. We then show how we can relax the condition to the case when the coaction is not injective and the Hopf C^*-algebra may be the full one.

In Section 3, we give some applications of the main theorem. In particular, we show that we don't need to assume amenability in the imprimitivity

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theorem in [7, 5.2] (see [7, 5.7(a)]). Moreover, we also prove an imprimitivity type theorem for crossed products of coactions by discrete type Kac C^*-algebras. Finally, we give an application to the case of discrete group coactions.

1. C^*-basic constructions for non-unital C^*-algebras

We would like to start with a slightly more general version of Watatani's C^*-basic construction (see [14]) in the case when the C^*-algebras may not be unital. Actually, it is just an application of the original version by Watatani. Let B be a C^*-algebra and let A be a C^*-subalgebra of B that contains an approximate unit of B. Let E be a faithful conditional expectation from B to A. In this case, we have the following easy lemma.

Lemma 1.1: A has a unit if and only if B has. Moreover, if it is the case, then the unit of B is in A.

As in [14, 2.1], we define an A-valued inner product on B by $\langle b, c\rangle = E(b^*c)$. Since E is a faithful conditional expectation, the inner product $\langle\cdot,\cdot\rangle$ makes B into a pre-Hilbert A-module. Let $\mathcal{F}$ be the completion of B with respect to this inner product and let η be the canonical map from B to $\mathcal{F}$. Consider a *-homomorphism λ from B to $\mathcal{L}(\mathcal{F})$ defined by $\lambda(a)(\eta(b)) = \eta(ab)$ and an element $e_A \in \mathcal{L}(\mathcal{F})$ given by $e_A(\eta(b)) = \eta(E(b))$. Using the same argument as in [14, 2.1], we can show that they are well defined and e_A is clearly a projection in $\mathcal{L}(\mathcal{F})$. As in [14, 2.1.2], we define the C^*-basic construction as follows:

Definition 1.2: Let $C^*\langle B, e_A\rangle$ be the closed linear span of the set $\{\lambda(a)e_A\lambda(b) : a, b \in B\}$ and call it the (reduced) C^*-basic construction.

Note that in [14], there are two kinds of C^*-basic construction but since they are isomorphic, we only consider one of them here. It is clear that $C^*\langle B, e_A\rangle = \mathcal{K}(\mathcal{F})$ (where $\mathcal{K}(\mathcal{F})$ is the closed linear span of the finite rank operators on $\mathcal{F}$). In the case when the C^*-algebras are unital, we have the following result from [14, 2.2.11]:

Theorem 1.3: (Watatani) Let B be a unital C^*-algebra and let A be a C^*-subalgebra of B that contains the unit of B. Let E be a faithful conditional expectation from B to A. If B acts on a Hilbert space H faithfully (and non-degenerately) and e is a projection on H such that

(i) $ebe = E(b)e$ for all $b \in B$ and
(ii) the map that sends $a \in A$ to $ae \in \mathcal{L}(H)$ is injective,
then the norm closure of BeB is isomorphic to $C^*\langle B, e_A\rangle$ canonically.

We would like to give a similar result in the non-unital case. In the remainder of this section, we assume that B is non-unital. Let $B^1 = B \oplus \mathbf{C}\cdot\mathbf{1}$ be the unitalisation of B and let A^1 be the C^*-subalgebra of B^1 generated by A and 1. Define a map E^1 from B^1 to A^1 by $E^1(x + \mu 1) = E(x) + \mu 1$. Then we have:

Proposition 1.4: E^1 is a faithful conditional expectation from B^1 to A^1.
Proof: We first show that $\| E^1 \| = 1$. Note that $A^1 \subseteq M(A) \subseteq M(B)$ (since A contains an approximate unit of B). Then $\| E^1(x + \mu 1) \|_{M(A)} = sup\{\| E^1(x + \mu 1)y \| : y \in A, \| y \| \leq 1\} = sup\{\| E(xy + \mu y) \| : y \in A, \| y \| \leq 1\} \leq sup\{\| xy + \mu y \| : y \in A, \| y \| \leq 1\} \leq \| x + \mu 1 \|_{M(B)}$. Moreover, since $E^1(1) = 1$, $\| E^1 \| = 1$. Let $u_i \in A$ be an approximate unit of B. Then for any $b \in B, \mu \in \mathbf{C}$, $(b + \mu u_i)^*(b + \mu u_i) \geq 0$. Since E is positive and of norm 1, $E(u_i^2) \leq 1$. Hence $E(b^*b)+ \mid \mu \mid^2 \cdot 1 + E(\overline{\mu} u_i b + \mu b^* u_i) \geq E((b + \mu u_i)^*(b + \mu u_i)) \geq 0$. Now the left hand side of the inequality converges to $E^1((b + \mu 1)^*(b + \mu 1))$ in norm. Hence, E^1 is positive. Now assume that $E^1((x + \mu 1)^*(x + \mu 1)) = 0$ and suppose that μ is non-zero. Then for any $a \in A$, $E((xa+\mu a)^*(xa+\mu a)) = 0$ and so $(xa+\mu a) = 0$ because E is faithful. Let $p = E(-x/\mu) \in A$. Then $pa = a$ for all $a \in A$ which contradicts the assumption that B is non-unital (see Lemma 1.1). Thus $\mu = 0$ and so $x = 0$ as well (as E is faithful).

Thus, we can construct $\mathcal{F}^1$ (the Hilbert A^1-module arising from E^1; see the paragraph after Lemma 1.1) and the C^*-basic construction for E^1. Let λ^1 be the canonical map from B^1 to $\mathcal{L}(\mathcal{F}^1)$ and e_A^1 be the element in $\mathcal{L}(\mathcal{F}^1)$ that corresponds to E^1. Now since B is a subalgebra of B^1 and E^1 extends E, $\mathcal{F}$ is a Hilbert A^1-submodule of $\mathcal{F}^1$ (note that $B \cdot A^1 \subseteq B$). It is clear that $e_A^1(\mathcal{F}) \subseteq \mathcal{F}$ and its restriction to $\mathcal{F}$ is e_A. Let $\mathcal{K}(\mathcal{F}^1)_0$ be the closed linear span in $\mathcal{L}(\mathcal{F}^1)$ of the set $\{\theta^1_{x,y} : x, y \in \eta(B)\}$ (where $\theta^1_{x,y}(z) = x\langle y, z\rangle$ and let η be the canonical map from B^1 to $\mathcal{F}^1$). For any $T \in \mathcal{K}(\mathcal{F}^1)_0$, $T(\mathcal{F}) \subseteq \mathcal{F}$ (since it is true for each $\theta^1_{x,y}$). Define a map Ψ from $\mathcal{K}(\mathcal{F}^1)_0$ to $\mathcal{L}(\mathcal{F})$ by $\Psi(T) = T \mid_{\mathcal{F}}$ (the restriction of T). Then $\| \Psi \| \leq 1$ and $\Psi(\mathcal{K}(\mathcal{F}^1)_0) \subseteq \mathcal{K}(\mathcal{F})$. It is clear that Ψ is a surjective *-homomorphism from $\mathcal{K}(\mathcal{F}^1)_0$ to $\mathcal{K}(\mathcal{F})$. Let u_i be an approximate unit for B. Then for any $T \in \mathcal{K}(\mathcal{F}^1)_0$, $T(\eta(u_i))$ will

converge to $T(\eta(1))$ (since it is true for $\theta^1_{x,y}$ and $\| \eta(u_i) \| \leq 1$). Hence, if $\Psi(T) = 0$, then $T(\eta(B)) = 0$ and so $T(\eta(B^1)) = 0$ which implies that $T = 0$. Therefore, we obtain the following:

Proposition 1.5: Let $\mathcal{K}(\mathcal{F}^1)_0$ be the closed linear span of the set $\{\theta^1_{x,y} : x, y \in \eta(B)\}$ in $\mathcal{L}(\mathcal{F}^1)$. Then $\mathcal{K}(\mathcal{F}^1)_0 \cong \mathcal{K}(\mathcal{F})$. Consequently, $C^*\langle B, e_A\rangle$ is isomorphic to the closed linear span of the set $\{\lambda^1(b)e^1_A\lambda^1(b') : b, b' \in B\}$ in $C^*\langle B^1, e^1_A\rangle$.

We can now obtain the result that we want:

Theorem 1.6: Let B be a C^*-algebra and let A be a C^*-subalgebra of B that contains an approximate unit of B. Let E be a faithful conditional expectation from B to A. If B acts on a Hilbert space H faithfully (and non-degenerately) and e is a projection on H such that
(1) $ebe = E(b)e$ for all $b \in B$ and
(2) the map that sends $a \in A$ to $ae \in \mathcal{L}(H)$ is injective,
then the norm closure of BeB is canonically isomorphic to $C^*\langle B, e_A\rangle$.
Proof: If B is unital, then it is just Theorem 1.3. Therefore, we assume that B is non-unital. Since B is represented faithfully on H, so is $M(B)$ and hence so is B^1. We would like to use Theorem 1.3. First of all, $e(b + \mu 1)e = E(b)e + \mu e = E^1(b + \mu 1)e$ and so condition (i) of Theorem 1.3 holds. Secondly, since A contains an approximate unit of B, $M(A)$ is represented non-degenerately on H through a map ϕ. Now let $K = eH$. Then, because of condition (1), condition (2) means that A is represented faithfully and non-degenerately on K (since A contains an approximate unit of B) and hence so is $M(A)$. Now it is clear that this representation of $M(A)$ on K is the restriction of ϕ. Hence the map that sends $a \in A^1$ to ae is injective. Now applying Theorem 1.3 to (B^1, A^1, E^1), we know that the closure of B^1eB^1 is isomorphic to $C^*\langle B^1, e^1_A\rangle$ under an isomorphism ψ which sends beb' to $\lambda^1(b)e^1_A\lambda^1(b')$. Hence the restriction of ψ will map the closure of BeB to $\{\lambda^1(b)e^1_A\lambda^1(b') : b, b' \in B\}$ which is isomorphic to $C^*\langle B, e_A\rangle$ by Proposition 1.5.

2. Morita equivalences between fixed point algebras and crossed products

In this section, we will prove a result about Morita equivalences between fixed point algebras and crossed products for a special kind of coaction of

compact quantum groups (Theorem 2.13). We first prove the restricted case when $S = S_V$ (see [1]) for a regular multiplicative V of compact type on a Hilbert space H_V such that there exists a faithful Haar state φ on S and the coaction is injective and effective in the following sense.

Definition 2.2: A coaction ϵ on B by S is said to be effective if $\epsilon(B)\cdot(B\otimes 1)$ is dense in $B \otimes S$.

Let B be a C^*-algebra with an injective coaction ϵ by S and let B^ϵ be the fixed point algebra. Let $E = (id \otimes \varphi)\epsilon$. It is not hard to show that E is a faithful conditional expectation from B to B^ϵ. Moreover, B^ϵ contains an approximate unit of B by the following easy lemma:

Lemma 2.1: Let B and D be C^*-algebras and ϕ be a state on D. If w_i is an approximate unit of $B \otimes D$, then $(id \otimes \phi)(w_i)$ is an approximate unit for B.

Note that any dual coaction is effective since comultiplications are effective. From now on, we can use the same lines of proof as in [7, Section 5] to deduce Theorem 2.7. However, for completeness, we will repeat the arguments here.

Lemma 2.3: $(id \otimes id \otimes \varphi)(V_{12}V_{13}) = (id \otimes id \otimes \varphi)(V_{13})$.
Proof: Since φ is the Haar state, $(id\otimes\varphi)\delta((\omega\otimes id)V) = (\omega\otimes id\otimes\varphi)(V_{13})$ for all $\omega \in \mathcal{L}(H_V)_*$. Now the lemma follows from the fact that $\mathcal{L}(H_V)_*$ separates points of $\mathcal{L}(H_V)$.

Lemma 2.4: Let B be faithfully represented on a Hilbert space H. Regard ϵ as an injective map from B to $\mathcal{L}(H \otimes H_V)$ and let $e = 1 \otimes p$ (where $p = (id \otimes \varphi)(V) \in \mathcal{L}(H_V)$). Then ϵ and e will satisfy the two conditions of Theorem 1.6.
Proof: Since for any $a \in B^\epsilon$, $\epsilon(a) = a \otimes 1$, condition (2) of Theorem 1.6 is clear. For condition (1), observe that $(1 \otimes p)\epsilon(b)(1 \otimes p) = (id \otimes id \otimes \varphi \otimes \varphi)(V_{23}(\epsilon(b) \otimes 1 \otimes 1)V_{24}) = (id \otimes id \otimes \varphi \otimes \varphi)(((id \otimes \delta_V)\epsilon(b) \otimes 1)V_{23}V_{24})$ for all $b \in B$. Thus, using Lemma 2.3, $(1 \otimes p)\epsilon(b)(1 \otimes p) = (id \otimes id \otimes \varphi \otimes \varphi)(((id \otimes \delta_V)\epsilon(b) \otimes 1)V_{24}) = [(id \otimes id \otimes \varphi)((id \otimes \delta_V)\epsilon(b))](1 \otimes p) = (id \otimes \varphi)(\epsilon(b)) \otimes p$. Hence we proved the lemma.

Lemma 2.5: The set $P = \{(id \otimes \varphi \cdot s)(V) : s \in A_V\}$ is a dense subset of $\hat{S}_V$.

Proof: We first note that because $p \cdot \hat{S}_V = \mathbf{C} \cdot p$, $p \in \hat{S}_V$ (where $p = (id \otimes \varphi)(V)$). Moreover, if $s = (\omega \otimes id)(V) \in A_V$, then $(id \otimes \varphi)((1 \otimes s)V) = (\omega \otimes id \otimes \varphi)(V_{13}V_{23}) = (id \otimes \varphi)(\omega \otimes id \otimes id)((\hat{\delta}_V \otimes id)V) = (\omega \otimes id)\hat{\delta}_V(p)$. Note that $\hat{\delta}_V(p)(x \otimes 1) \in \hat{S}_V \otimes \hat{S}_V$ (for any $x \in \hat{S}_V$) and so $(\omega \otimes id)\hat{\delta}_V(p) \in \hat{S}_V$. Thus we have showen that P is a subset of $\hat{S}_V$. Let $t \in S_V$ be such that $(\varphi \cdot s)(t) = 0$ for all $s \in A_V$. Then $\varphi(t^*t) = 0$ (as A_V is dense in S_V). Because φ is faithful, $P' = \{\varphi \cdot s : s \in A_V\}$ separates points of S_V. Hence P' is $\sigma(S_V^*, S_V)$-dense in S_V^*. Therefore, for any $f \in S_V^*$, there exists a net s_i in A_V such that $\varphi \cdot s_i$ converges to f weakly. Note that $g(L_V(h)) = h(\rho_V(g))$ for all $h \in \hat{S}_V^*$ and $g \in S_V^*$ and that $L_V(\hat{S}_V^*)$ is a dense subset of S_V (since $1 \in S_V$). Hence for any $\nu \in \mathcal{L}(H_V)_*$, there exists a net a_i in $P = \rho_V(P')$ such that $h(a_i)$ converges to $h(\rho_V(\nu))$ for any $h \in \hat{S}_V^*$. Therefore, the $\sigma(\hat{S}_V, \hat{S}_V^*)$-closure of P will contain $\hat{A}_V$ and so P is norm dense in $\hat{S}_V$ (because P is a convex subset of $\hat{S}_V$).

Lemma 2.6: Let the notation be the same as in Lemma 2.4. If, in addition, ϵ is effective, then the linear span, T, of $\{\epsilon(a)(1 \otimes p)\epsilon(b) : a, b \in B\}$ is norm dense in $B \times_{\epsilon,r} \hat{S}_V$

Proof: We first note that T is a subset of $B \times_{\epsilon,r} \hat{S}_V$. Since ϵ is a coaction, $(1 \otimes p)\epsilon(b) = (id \otimes id \otimes \varphi)((\epsilon \otimes id)\epsilon(b)V_{23})$. Therefore, $\epsilon(a)(1 \otimes p)\epsilon(b) = (id \otimes id \otimes \varphi)((\epsilon \otimes id)((a \otimes 1)\epsilon(b))V_{23})$. Since ϵ is effective, elements of the form $(id \otimes id \otimes \varphi)((\epsilon \otimes id)(c \otimes s)V_{23})$ ($c \in B$ and $s \in S_V$) can be approximated in norm by elements in T. Note that $(id \otimes id \otimes \varphi)((\epsilon \otimes id)(c \otimes s)V_{23}) = \epsilon(c)(1 \otimes (id \otimes \varphi \cdot s)(V))$. Hence by Lemma 2.5, T is norm dense in $B \times_{\epsilon,r} \hat{S}_V$.

We can now state and prove the main theorem in this section.

Theorem 2.7: Let B be a C^*-algebra with an injective and effective coaction ϵ by S_V. Let B^ϵ be the fixed point algebra of ϵ. Then B^ϵ is strongly Morita equivalent to the reduced crossed product $B \times_{\epsilon,r} \hat{S}_V$ ($= B \times_{\epsilon,max} \hat{S}_V$).

Proof: By Lemma 2.4, Theorem 1.6 and the fact that the Hilbert-B^ϵ-module $\mathcal{F}$ (defined by the conditional expectation) is full, B^ϵ is strongly Morita equivalent to the closure of the linear span of the set $\{\epsilon(a)(1 \otimes p)\epsilon(b) : a, b \in B\}$ which, by Lemma 2.6, equals $B \times_{\epsilon,r} \hat{S}_V$.

We are now going to relax the conditions on ϵ and S. We first deal with the case when ϵ is not injective. As in [6, 2.17], we consider $I = Ker(\epsilon)$. Then there is a coaction ϵ' on $A = B/I$ given by $\epsilon'(q(b)) = (q \otimes id)\epsilon(b)$ for any $b \in B$ (where q is the canonical quotient from B to A). It is easily seen

that if ϵ is effective, then so is ϵ'. Note that $(id \otimes \varphi)\epsilon(B) = B^\epsilon$ no matter ϵ is injective or not. We first show the following lemma:

Lemma 2.8: With the notation above, $A^{\epsilon'}$ is isomorphic to B^ϵ.
Proof: Note that $A^{\epsilon'} = (id \otimes \varphi)\epsilon'(q(B)) = (q \otimes \varphi)\epsilon(B) = q(B^\epsilon)$. Now if $b \in B^\epsilon$ be such that $q(b) = 0$, then $b \in I = Ker(\epsilon)$ and $\epsilon(b) = b \otimes 1$ which implies that $b = 0$. Hence, the restriction of q on B^ϵ is injective and is therefore an isomorphism.

Proposition 2.9: Let everything be the same as Theorem 2.7 except that we don't assume ϵ to be injective. Then B^ϵ is strongly Morita equivalent to the reduced crossed product $B \times_{\epsilon,r} \hat{S}_V$.
Proof: Since V is of compact type, it is automatically amenable. Therefore, by [6, 2.19(a)], we have $A \times_{\epsilon',r} \hat{S}_V = B \times_{\epsilon,r} \hat{S}_V$ (where A and ϵ' be as defined above). Therefore, B^ϵ is strongly Morita equivalent to $B \times_{\epsilon,r} \hat{S}_V$ by Theorem 2.7.

We now turn to the case when $S = (S_V)_p$ (see [6]).

Lemma 2.10: Let $\varphi_p = \varphi \circ L_V$. Then φ_p is an invariant state on $(S_V)_p$ (in the sense that $(id \otimes \varphi_p)\delta_p = \varphi_p \cdot 1$).
Proof: Let p be the minimum central projection in $\hat{S}_V$ that corresponds to φ. Let χ be the injective norm decreasing algebraic homomorphism from $(S_V)_p^*$ to $M(\hat{S}_V)$ as given in [6, A6]. Then it is clear that $\chi(\varphi_p) = p$ and hence φ_p is a central minimum projection in $(S_V)_p^*$. This means that $(f \otimes \varphi_p)\delta_p = f(1)\varphi_p$ for any $f \in (S_V)_p^*$ (note that $\delta_p(1) = 1 \otimes 1$).

Let B be a C^*-algebra with coaction ϵ'' by $(S_V)_p$. As in [6, 2.14], we consider $\epsilon = (id \otimes L_V)\epsilon''$. It is clear that if ϵ'' is effective, then so is ϵ. Note that we also have $(id \otimes \varphi_p)\epsilon''(B) = B^{\epsilon''}$. Now $B^{\epsilon''} = (id \otimes \varphi_p)\epsilon''(B) = (id \otimes \varphi)\epsilon(B) = B^\epsilon$. Moreover, we recall from [6, 2.14] that $B \times_{\epsilon'',r} (\hat{S}_V)_p$ is isomorphic to $B \times_{\epsilon,r} \hat{S}_V$. These, together with Proposition 2.9, proved the following generalisation of Theorem 2.7:

Proposition 2.11: Let B be a C^*-algebra with an effective coaction ϵ by $S = S_V$ or $(S_V)_p$. Let B^ϵ be the fixed point algebra of ϵ. Then B^ϵ is strongly Morita equivalent to the reduced crossed product $B \times_{\epsilon,r} \hat{S}$ $(= B \times_{\epsilon,max} \hat{S})$.

Now we would like to further generalise S. Let S be a compact quantum group (i.e. an unital Hopf C^*-algebra with comultiplication δ such that both

$\delta(S)(S\otimes 1)$ and $\delta(S)(1\otimes S)$ are dense in $S\otimes S$). By [11], S has a Haar state ϕ. Let (H,π) be the GNS representation corresponding to ϕ and V be the multiplicative unitary on H as defined in [1, 1.2(4)]. Then π is a surjective map from S to S_V. Let φ be the Haar state on S_V (which equals the state defined by ϕ). Then φ is faithful by the following lemma.

Lemma 2.12: Let A be a compact quantum group with a Haar state ψ. Then ψ is a faithful state if and only if the GNS representation (H,π) corresponding to ψ is faithful.
Proof: Suppose that (H,π) is faithful. Let $N=\{x\in A:\psi(x^*x)=0\}$. Then it is clear that $N=\{x\in A:\psi(yx)=0 \text{ for all } y\in A\}$. By [15, 5.6(6)], $\psi(ab)=\psi(b(f_1*a*f_1))$ for all $a\in\mathcal{A}$. Now since $\mathcal{A}$ is a dense subalgebra of A and $f_1*\mathcal{A}*f_1=\mathcal{A}$, $N=\{x\in A:\psi(xy)=0 \text{ for all } y\in A\}$. Hence N is an ideal of A and so $N=Ker(\pi)=0$. Thus ψ is faithful.

Let B be a C^*-algebra with coaction ϵ'' by S and let $\epsilon=(id\otimes\pi)\epsilon''$. Then ϵ is a coaction on B by S_V. By the same argument as in Proposition 2.11, we have $B^\epsilon=B^{\epsilon''}$. Therefore, if we define the reduced crossed product of ϵ'' to be the reduced crossed product of ϵ, then we have the following:

Theorem 2.13: Let B be a C^*-algebra with an effective coaction ϵ by a compact quantum group S. Then B^ϵ is strongly Morita equivalent to the reduced crossed product $B\times_{\epsilon,r}\hat{S}$.

Note that the assumption of effectiveness cannot be removed. For example, if we consider ϵ to be the trivial coaction on B (i.e. $\epsilon(b)=b\otimes 1$), then the fixed point algebra is B itself while the reduced crossed product is $B\otimes\hat{S}$ which are clearly not Morita equivalent (e.g. when $B=\mathbf{C}$ and S is non-trivial).

Remark 2.14: (a) We was told that van Daele and Zhang have recently proved a similar result to Theorem 2.13 (but in a purely algebraic framework) for algebraic quantum groups (which can be regarded as a generalization of both compact quantum groups and discrete quantum groups). For a reference, we refer the readers to [12].
(b) By the proof of Theorem 2.7, we know that in general (i.e. when ϵ is not effective), B^ϵ is strongly Morita equivalent to the ideal $I=\{\epsilon(a)(1\otimes p)\epsilon(b): a,b\in B\}$ of $B\times_{\epsilon,r}\hat{S}_V$ (I is an ideal since p is a central minimum projection). Consequently, if the reduced crossed product satisfies some properties that

are preserved under taking ideal and under Morita equivalence, then so does the fixed point algebra. In particular, we have the following corollary.

Corollary 2.15: Let B be a C^*-algebra with a coaction (not necessary effective) ϵ by a compact quantum group S. Then
(a) if $B \times_{\epsilon,r} \hat{S}$ is nuclear (this is the case if B is, see [6, 3.4]), so is B^ϵ;
(b) if $B \times_{\epsilon,r} \hat{S}$ is liminal (resp. postliminal), then so is B^ϵ;
(c) if $B \times_{\epsilon,r} \hat{S}$ is simple, then it is strongly Morita equivalent to B^ϵ and so B^ϵ is simple.

Finally, we would like to apply Theorem 2.7 to the case of coactions by discrete groups. We recall from [4, 2.6], that if A is a C^*-algebra with "reduced" coaction ϵ by a discrete group G (i.e. an injective and non-degenerate coaction by $C_r^*(G)$), then $A = \overline{\bigoplus_{t\in G} A_t}$ (with A_e being the fixed point algebra of ϵ). Let ψ be the Haar state on $C_r^*(G)$ and λ_t be the canonical image of $t \in G$ in $C_r^*(G)$. Let $\psi_t = \psi \cdot \lambda_{t^{-1}}$. Then $A_t = (id \otimes \psi_t)\epsilon(A)$. Recall from [5, Section 5] that ϵ is said to be full if $\overline{A_t \cdot A_{t^{-1}}} = A_e$ for all $t \in G$ (or equivalently, $\overline{A_r \cdot A_s} = A_{rs}$ for all $r, s \in G$).

Lemma 2.16: ϵ is effective if and only if it is full.
Proof: Suppose that ϵ is effective. Then by acting $(id \otimes \psi_t)$ on both sides of the equation $\overline{\epsilon(A) \cdot (A \otimes 1)} = A \otimes C_r^*(G)$, we have $\overline{A_t \cdot A} = A$. Thus by applying $(id \otimes \psi_{-t})\epsilon$ again, we obtain $\overline{A_t \cdot A_{t^{-1}}} = A_e$. Conversely, suppose that ϵ is full. It is required to show that $\overline{\epsilon(A) \cdot (A \otimes 1)} \supseteq A_r \otimes \lambda_s$. Actually, it suffices to show that $(A_s \otimes \lambda_s) \cdot (A_{s^{-1}r} \otimes 1) \supseteq A_r \otimes \lambda_s$. However, it is clear from the definition of full coaction that this relation holds.

Now using Theorem 2.7, we can obtain as a corollary [5, 5.2] (which stated that the fixed point algebra and the crossed product are Morita equivalent if the coaction is full).

3. An application of the main theorem

In this section, we will use the main result in Section 2 to prove a imprimitivity type theorem for crossed products of coactions by discrete quantum groups. We first recall some basic definition from [7, Section 4].

Definition 3.1: Let U and V be regular multiplicative unitaries on the Hilbert spaces H and K respectively. $X \in \mathcal{L}(K \otimes H)$ is said to be a U-V-birepresentation if X is a representation of V as well as a corepresentation of U.

We recall from [7, 3.9] that a U-V-birepresentation X will induce a Hopf *-homomorphism $L_{X''}$ from $(S_U)_p$ to $M[(S_V)_p]$ as well as a Hopf *-homomorphism $\rho_{X'}$ from $(\hat{S}_V)_p$ to $M[(\hat{S}_U)_p]$. Therefore, we have a coaction $\epsilon_X = (id \otimes \rho_{X'})\hat{\delta}_V$ on $(\hat{S}_V)_p$ by $(\hat{S}_U)_p$.

Definition 3.2: Let U, V, W be regular multiplicative unitaries.
(a) W is said to be a sub-multiplicative unitary of V if there exists a V-W-birepresentation Y such that $L_{Y''}((S_V)_p) = (S_W)_p$.
(b) U is said to be a quotient of V if there exists a U-V-birepresentation X such that $\rho_{X'}((\hat{S}_V)_p) = (\hat{S}_U)_p$.
(c) Let U, V and W be of discrete type such that U is a quotient of V through X and W is a submultiplicative unitary of V through Y. Then W is said to be normal if $\rho_{Y'}$ is an isomorphism from $(\hat{S}_W)_p$ to the fixed point algebra of $\hat{\epsilon}_X$ in $(\hat{S}_V)_p$.

The above terminologies come from the case of locally compact groups. If H and G are two locally compact groups, then H is a quotient of G if and only if $C^*(H)$ is a quotient of $C^*(G)$ as Hopf C^*-algebras. Moreover, H is a subgroup of G if and only if $C_0(H)$ is a quotient of $C_0(G)$ (as Hopf C^*-algebras). A normal subgroup of a discrete group will certainly satisfy condition 3.2(c) (see [7,A2]). Using Proposition 2.11, we can now give a positive answer to the remark in [7, 5.7(a)] (i.e. we don't need to assume the amenability for U in the imprimitivity theorem). More precisely, we have:

Theorem 3.3: Let U, V and W be regular multiplicative unitaries of discrete type such that U comes from a discrete Kac algebra. If W is a normal submultiplicative unitary of V with quotient U, then $(\hat{S}_W)_p$ is strongly Morita equivalent to $(\hat{S}_V)_p \times_{\epsilon',r} S_U$.

Moreover, we can have the following version of the imprimitivity theorem for crossed products: Let U, V and W be multiplicative unitaries that come from discrete Kac algebras such that W is a normal sub-multiplicative unitary of V (through a V-W-bi-representation Y) with quotient U (through a U-V-bi-representation X). Furthermore, suppose that W is amenable (note that all U, V and W are co-amenable in this case).

Theorem 3.4: Let U, V and W be as given in the previous paragraph. Let A be a C^*-algebra with an injective and non-degenerate coaction ϵ by $S_V = (S_V)_p$ and let $\hat{\epsilon}$ be the dual coaction on the full crossed product $A \times_\epsilon \hat{S}_V$

(see [7, 1.12]). Suppose that ϵ' and ϵ'' are the coactions on A and $A \times_\epsilon \hat{S}_V$ by S_W and $(\hat{S}_U)_p$ induced from ϵ and $\hat{\epsilon}$ respectively (i.e. $\epsilon' = (id \otimes L_{Y''}\epsilon$ and $\epsilon'' = (id \otimes \rho_{X'})\hat{\epsilon})$. Then $A \times_{\epsilon'} \hat{S}_W$ is strongly Morita equivalent to $(A \times_\epsilon \hat{S}_V) \times_{\epsilon'',r} S_U$.

We need several lemmas to prove this theorem.

Lemma 3.5: If ϵ is an injective and non-degenerate coaction on a C^*-algebra A by S_V, then the induced coaction $\epsilon' = (id \otimes L_{Y''})\epsilon$ on A by S_W is also injective and non-degenerate.
Proof: Let E_V be the co-identity on $S_V = (S_V)_p$. Then for any $a \in A$, $\epsilon((id \otimes E_V)\epsilon(a) - a) = 0$ and so $(id \otimes E_V)\epsilon(a) = a$. Now if E_W is the co-identity on S_W, then $E_V = E_W \circ L_Y$ by [7, 3.2(d)]. Therefore, $(id \otimes E_W)\epsilon'(a) = a$ and so ϵ' is injective. ϵ' is non-degenerate since L_Y is surjective.

Lemma 3.6: Let B and C be C^*-algebras and ϕ a non-degenerate *-homomorphism from B to C with kernel I. Let $\overline{\phi}$ be the extension of ϕ from $M(B)$ to $M(C)$. Then $Ker(\overline{\phi}) = \{m \in M(B) : m \cdot B, B \cdot m \subseteq I\}$.
Proof: Let Q be the quotient map from B to B/I and ψ be the canonical non-degenerate monomorphism from B/I to C. Then $\overline{\phi} = \overline{\psi} \circ \overline{Q}$. Now the lemma follows from the easy facts that $Ker(\overline{Q}) = \{m \in M(B) : m \cdot B, B \cdot m \subseteq I\}$ and that $\overline{\psi}$ is injective.

Lemma 3.7: Let B and C be two C^*-algebras with coactions ϵ_B and ϵ_C respectively by a Hopf C^*-algebra S. Suppose that S is C^*-exact (as a C^*-algebra). If ψ is a non-degenerate equivariant *-homomorphism from B to C, then $Ker(\psi)$ is a weakly invariant ideal of A (in the sense of [8, 3.14]).
Proof: Let $I = Ker(\psi)$ and q be the canonical quotient from B to B/I. Since S is C^*-exact, $Ker(q \otimes id) = I \otimes S$. Hence $Ker(\psi \otimes id) = Ker(\tilde{\psi} \circ q \otimes id) = I \otimes S$ (where $\tilde{\psi}$ is the canonical injection from B/I to C). Therefore, by Lemma 3.6, $Ker(\overline{\psi \otimes id}) = \{m \in M(B \otimes S) : m \cdot (B \otimes S), (B \otimes S) \cdot m \subseteq I \otimes S\}$. Now for any $x \in I$, $(\overline{\psi \otimes id})\epsilon_B(x) = \epsilon_C(\psi(x)) = 0$. Thus, $\epsilon_B(x) \in \tilde{M}(B \otimes S) \cap Ker(\overline{\psi \otimes id}) \subseteq \tilde{M}(I \otimes S)$ (by [2, 1.4]).

Let D be a C^*-algebra with an injective coaction δ by a Hopf C^*-algebra that is defined by a regular multiplicative unitary. Let (C, j_D, ν) be the full crossed product of δ (see [6]). Then j_D is injective because the canonical map from D to the reduced crossed product is injective and the reduced representation is covariant.

Stimulated by [9, 4.3], we have the following lemma.

Lemma 3.8: Let B be a C^*-algebra with an injective and non-degenerate coaction ϵ by $S = S_{V'}$ where V' is a co-amenable irreducible multiplicative unitary on a Hilbert space H. Let D be a C^*-algebra with an injective coaction δ by $\hat{S} = \hat{S}_{V'}$. If ϕ is an (non-degenerate) equivariant *-homomorphism from $B \times_{\epsilon,r} \hat{S}$ to D such that the restriction of ϕ on B (considered as a subalgebra of $M(B \times_{\epsilon,r} \hat{S})$) is injective, then ϕ is injective.
Proof: Since ϕ is equivariant, by [6, 3.9], it induces a map Φ from $(B\times_{\epsilon,r}\hat{S})\times_{\hat{\epsilon}} S$ to $D\times_\delta S$ such that $\Phi\circ j = j_D\circ\phi$ and $\Phi\circ\mu = \nu$ (where $((B\times_{\epsilon,r}\hat{S})\times_{\hat{\epsilon}} S, j, \mu)$ and $(D \times_\delta S, j_D, \nu)$ are the full crossed products of $\hat{\epsilon}$ and δ respectively). It is not hard to show that Φ is equivariant with respect to the dual coactions defined in [7, 1.12]. Now since V' is a co-amenable irreducible multiplicative unitary, $(B \times_{\epsilon,r} \hat{S}) \times_{\hat{\epsilon}} S = (B \times_{\epsilon,r} \hat{S}) \times_{\hat{\epsilon},r} S \cong B \otimes \mathcal{K}(H)$ (by [1, 7.5]). It is also clear that the dual coactions defined in [7, 1.12] and in [1, Section 7] are identical in this case. Now by the proof of [1, 7.5], for any $b \in B$, $\pi_L(b) \otimes 1$ ($\in B \times_{\epsilon,r} \hat{S} \times_{\hat{\epsilon},r} S$) corresponds to $\pi_R(b) = (id \otimes R)\epsilon(b)$ ($\in B \otimes \mathcal{K}(H)$) under the isomorphism. Regard Φ as a map from $B \otimes \mathcal{K}(H)$ to $D \times_\delta S$. Then we have $\Phi(\pi_R(b)) = j_D(\phi(b))$. Let $J = Ker(\Phi)$. J is weakly invariant by Lemma 3.7 (note that S is nuclear by [6, 3.6]). Moreover, by the proof of [8, 3.15], $J = I \otimes \mathcal{K}(H)$ for some weakly invariant ideal I of B. Now for any $x \in I$, $\epsilon(x) \in \tilde{M}(I \otimes S)$. Since R is an isomorphism from S to $\hat{S}_{\tilde{V}} \subseteq \mathcal{L}(H)$, $\pi_R(x) \in \tilde{M}(I \otimes \hat{S}_{\tilde{V}}) \subseteq Ker(\overline{\Phi})$ (by Lemma 3.6). Hence $j_D(\phi(x)) = \Phi(\pi_R(x)) = 0$ and so $\phi(x) = 0$ (note that j_D is injective as δ is injective). Therefore, from the hypothesis, $I = 0$ and so Φ is injective. Using the facts that j is injective (the dual coaction $\hat{\epsilon}$ is injective) and that $\Phi \circ j = j_D \circ \phi$, we showed that ϕ is also injective.

Now we can give a proof for Theorem 3.4.

Proof: (Theorem 3.4) Let $(A \times_\epsilon \hat{S}_V, j, \mu)$ be the full crossed product of ϵ. Then $(j \otimes id)\epsilon'(a) = (id \otimes L_Y)((\mu \otimes id)(V')(j(a) \otimes 1)(\mu \otimes id)(V')^*) = ((\mu \otimes id)(Y')(j(a) \otimes 1)(\mu \otimes id)(Y')^*)$. Hence, $(j, \mu \circ \rho_{Y'})$ is a covariant pair for ϵ' and so there exists a non-degenerate *-homomorphism ϕ from $A \times_{\epsilon'} \hat{S}_W$ to $M(A \times_\epsilon \hat{S}_V)$. Moreover, since $1 \in (\hat{S}_V)_p$, $j(A) \subseteq A \times_\epsilon \hat{S}_V$. Hence $\phi(A \times_{\epsilon'} \hat{S}_W) \subseteq A \times_\epsilon \hat{S}_V$. Let φ be the Haar state on $(\hat{S}_U)_p$ and let $D = (A \times_\epsilon \hat{S}_V)^{\epsilon''}$. Then $D = (id \otimes \varphi)\epsilon''(A \times_\epsilon \hat{S}_V)$. Now for any $a \in A$ and $t \in (\hat{S}_V)_p$, $(id \otimes \varphi)\epsilon''(j(a)\mu(t)) = (id \otimes \varphi \circ \rho_{X'})((j(a) \otimes 1)(\mu \otimes id)\hat{\delta}_V(t)) = j(a)(\mu\otimes\varphi)\epsilon_{X'}(t)$. By the assumption that $\rho_{Y'}$ is an isomorphism from $(\hat{S}_W)_p$

to $(\hat{S}_V)_p^{\epsilon_{X'}}$, we have $\phi(A\times_{\epsilon'}\hat{S}_W) = D$. Next, we would like to show that ϕ is injective. Note that for any $a\in A$ and $s\in(\hat{S}_W)_p$, $\hat{\epsilon}(j(a)\mu(\rho_{Y'}(s))) = (j(a)\otimes 1)(\mu\otimes id)(\rho_{Y'}\otimes\rho_{Y'})(\hat{\delta}_W(s)) \in (A\times_\epsilon\hat{S}_V)^{\epsilon''}\otimes(\hat{S}_W)_p$ (since $(\hat{S}_W)_p$ is unital and $\rho_{Y'}$ is a Hopf *-monomorphism). Moreover, if u_i is an approximate unit of A, then $\hat{\epsilon}(j(u_i)) = j(u_i)\otimes 1$ is an approximate unit for $(A\times_\epsilon\hat{S}_V)^{\epsilon''}\otimes(\hat{S}_W)_p$. Hence $\hat{\epsilon}$ induces an injective coaction δ on $(A\times_\epsilon\hat{S}_V)^{\epsilon''}$ by $(\hat{S}_W)_p = \hat{S}_W$ (note that δ is injective since $(id\otimes\hat{E}_W)\delta = id$ with $\hat{E}_W$ being the co-identity of $(\hat{S}_W)_p$). Furthermore, ϕ is clearly injective on A since j is injective (note that ϵ is injective). Therefore, by Lemma 3.8, ϕ is injective. We are now going to use Proposition 2.11 to prove this theorem. It remains to show that ϵ'' is effective. Since $\rho_{X'}$ is surjective (by assumption), it suffices to show that $\hat{\epsilon}$ is effective. However, since $\hat{\delta}_V((\hat{S}_V)_p)\cdot((\hat{S}_V)_p\otimes 1) = (\hat{S}_V)_p\otimes(\hat{S}_V)_p$, it is easily seen that $\hat{\epsilon}$ is effective.

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